A CLASS OF EXACT SOLUTIONS OF THE IDEAL PLASTICITY EQUATIONS

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UDC 539.374: 539.374.1

A class of exact solutions is formulated for the ideal plasticity equations in the case of plane deformation. The solutions describe the plastic state of various wedges, notched half-planes, and domains in the form of funnels produced by detonation, the plastic state of domains having a horn configuration, etc. Many of the solutions have natural boundaries comprising envelopes of slip lines. The equations for the boundaries, slip lines, and stresses are presented in explicit form.

1. We consider the plane deformation of an ideally plastic medium. Let (x_1, x_2) be the Cartesian coordinates in the plane Ox_1x_2, (λ_1, λ_2) the characteristic coordinates, σ the first invariant of the stress tensor, φ the slope angle of the maximum principal stress relative to the axis Ox_1, and k a constant of the material. In this notation the limiting equilibrium equations for the medium have the form [1]:

$$\frac{1}{2k}\frac{\partial\sigma}{\partial\lambda_1}-\frac{\partial\varphi}{\partial\lambda_1}=0;\quad \frac{1}{2k}\frac{\partial\sigma}{\partial\lambda_2}+\frac{\partial\varphi}{\partial\lambda_2}=0;$$
$$\frac{\partial x_2}{\partial\lambda_1}=\operatorname{tg}\left(\varphi-\frac{\pi}{4}\right)\frac{\partial x_1}{\partial\lambda_1};\quad \frac{\partial x_2}{\partial\lambda_2}=\operatorname{tg}\left(\varphi+\frac{\pi}{4}\right)\frac{\partial x_1}{\partial\lambda_2}. \tag{1.1}$$

In the plane Ox_1x_2 we introduce polar coordinates (r, θ) and a new unknown function δ, which is the angle between the coordinate lines θ =const and λ_2=const. It follows from Eqs. (1.1) and the definition of δ that

$$\frac{1}{r}\frac{\partial r}{\partial\lambda_1}=\operatorname{ctg}\delta\frac{\partial\theta}{\partial\lambda_1};\quad \frac{1}{r}\frac{\partial r}{\partial\lambda_2}=-\operatorname{tg}\delta\frac{\partial\theta}{\partial\lambda_2}; \tag{1.2}$$
$$\varphi=\Phi_1(\lambda_1)-\Phi_2(\lambda_2),\quad \theta=\varphi-\delta-\frac{\pi}{4}=\Phi_1(\lambda_1)-\Phi_2(\lambda_2)-\delta-\frac{\pi}{4},$$

where $\Phi_1(\lambda_1)$, $\Phi_2(\lambda_2)$ are arbitrary functions. Eliminating r and θ from the system (1.2), we obtain one equation in δ:

$$\frac{\partial^2}{\partial\lambda_1\partial\lambda_2}\ln|\operatorname{tg}\delta|+\Phi_2'(\lambda_2)\frac{\partial}{\partial\lambda_1}\operatorname{tg}\delta-\Phi_1'(\lambda_1)\frac{\partial}{\partial\lambda_2}\operatorname{ctg}\delta=0.$$

We seek the solution of the latter equation in the class of functions $\tan\delta=\xi_1(\lambda_1)/\xi_2(\lambda_2)$. If $c_1\Phi_1+c_2\geq 0$; $c_1\Phi_2+c_3\geq 0$, where c_1, c_2, and c_3 are constants, then in the given class the solution has the form

$$\delta=\pm\operatorname{arctg}\sqrt{\frac{c_1\Phi_1+c_2}{c_1\Phi_2+c_3}}. \tag{1.3}$$

For $c_1=0$ the slip lines λ_1 and λ_2 are either two families of logarithmic spirals or two families of coordinate lines r=const, θ =const. This variant has been discussed in [1]. For $c_1\neq 0$ it may be assumed without loss of generality that $c_1=1$; $c_2=c_3=0$; $\Phi_1(\lambda_1)=\lambda_1^2=\mu_1$; $\Phi_2(\lambda_2)=\lambda_2^2=\mu_2$, where $\mu_1, \mu_2\geq 0$. After the substitution of (1.3) into (1.2) and (1.1) Eq. (1.1) is readily integrated:

$$\frac{\sigma}{2k}=\rho^2+\sigma^0;\quad \varphi=\rho^2\cos 2\nu+\frac{\pi}{2}+\varphi^0; \tag{1.4}$$

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 102–107, March-April, 1975. Original article submitted March 18, 1974.

$$r = \frac{2r^0}{\rho \sin 2\nu} e^{\pm\rho^2 \sin 2\nu}; \quad \theta = \rho^2 \cos 2\nu \mp \left(\frac{\pi}{2} - \nu\right) + \frac{\pi}{4} + \varphi^0;$$
$$\delta = \pm\left(\frac{\pi}{2} - \nu\right),$$

where (ρ, ν) are polar coordinates in the plane of the parameters (μ_1, μ_2); r^0 and φ^0 are constants reflecting the arbitrary choice of length scale and reference origin for the angle θ; and σ^0 is the additive constant pressure. Hereinafter we let $r^0=1$, $\varphi^0=0$, and $\sigma^0=0$.

The solution (1.4) is conveniently analyzed in the plane of the parameters (μ_1, μ_2); the coordinate lines μ_1, μ_2 in this plane are the slip lines in the physical plane Ox_1x_2, the slope angle of the radius ν =const relative to the μ_2 axis is equal to the angle of intersection of the slip line μ_1 in the physical plane with the radius θ =const, and, finally, the square of the radius ρ in the plane (μ_1, μ_2) is equal to the dimensionless compression $\sigma/2k$ at the corresponding points of the physical plane.

2. Let us take the upper signs in (1.4). In the single-sheeted plane Ox_1x_2 the solution (1.4) can exist only up to the branch line $\Delta = \partial(r, \theta)/\partial(\mu_1, \mu_2) = 0$. The line $\Delta = 0$ satisfies the equation $\sin 2\nu = 4\rho^2/(1+4\rho^4)$ and has two branches: $\rho^2 = {}^1/_2 \cot\nu$, $\rho^2 = {}^1/_2 \tan\nu$ (Fig. 1). It can be shown that in the plane Ox_1x_2 the branch $\rho^2 = {}^1/_2 \cot\nu$ is the envelope of the slip lines μ_1 and the turning line of the family μ_2. Analogously, the branch $\rho^2 = \frac{1}{2}\tan\nu$ is the envelope of the family of slip lines μ_2 and the turning line of the family μ_1. The increment of the angle θ in the plane Ox_1x_2 cannot be greater than 2π. In (1.4), therefore, we can only consider those parameters for which the increment of θ is less than 2π. Thus, considering the mapping of various (μ_1, μ_2) domains in which $\Delta \neq 0$ and the variation of θ is less than 2π, onto the plane Ox_1x_2, we obtain various integrals of Eqs. (1.1). These integrals can be interpreted as exact solutions of the corresponding boundary-value problems.

In the (μ_1, μ_2) plane we construct the lines $\theta = \theta^0$=const and analyze the particular solutions (1.4). It follows from (1.4) that the mappings of (μ_1, μ_2) domains symmetric about the bisector $\mu_1=\mu_2$ onto the plane Ox_1x_2 are symmetric about the line $\theta=0$. If $0 \le \theta^0 \le \pi/4$, the line $\theta = \theta^0$ is defined for $0 \le \nu \le \pi/4$; $\pi/4 + \theta^0 \le \nu \le \pi/2$; if $\theta^0 > \pi/4$, it is defined for $0 \le \nu < \pi/4$. For $\theta = \theta_0 > 0$, $0 \le \nu < \pi/4$, $r \to \infty$ the angle $\delta \to \pi/4$ and the pressure $\sigma \to \infty$; but if $0 < \theta^0 < \pi/4$; $\pi/4 + \theta_0 < \nu \leqslant \pi/2$, $r \to \infty$, then $\delta \to \pi/4 - \theta^0$ and $\sigma \to 0$. Moreover, $\left.\frac{d\mu_1}{d\mu_2}\right|_{\theta=\theta^0>0} \gtreqless 0$ for $\cos 2\nu \times \left(\rho^2 - {}^1/_2 \operatorname{ctg}\nu\right) \gtreqless 0$. We consider the flow corresponding in (μ_1, μ_2) to the domain $A_1A_3A_4A_6A_1$ (Fig. 1). The domain $A_1A_3A_4A_6A_1$ maps onto the symmetric wedge domain $B_1B_3B_4B_6B_1'$ in Fig. 2. The wedge angle is equal to $\pi/2$. The side $B_4B_3B_1$ is the envelope of the family of lines μ_1. In the part B_3B_1 ${}^1/_2 \cos^{-1}(1-\sqrt{5})/2 < \nu < \pi/2$ the curve $B_4B_3B_1$ is concave, and in the part B_4B_3 $\pi/4 \le \nu < {}^1/_2 \cos^{-1}(1-\sqrt{5})/2$ it is convex; as $r \to \infty$ the angle $\theta(B_1) \to \pi/4$. The sides of the wedge are acted upon by a constant tangential stress τ =k and a normal stress that tends to zero as $r \to \infty$. As $x_1 \to \infty$ the stressed state in the wedge interior tends to a uniform state, the slip lines tend to the straight lines $x_2 = \pm x_1$ +const, the angle $\varphi \to \pi/2$, and the compression $\sigma \to 0$.

Next we consider the flows against both sides of the curve A_4A_5 and those bounded by this curve and the line $\mu_2 = \mu_2^0$ = const; $0 < \mu_2^0 \le 1/2$. In the physical plane these flows have a horn configuration. The inner boundary of the flows is the envelope of the slip lines μ_1, and the outer boundary is one of the lines μ_1. As $\theta \to \infty$ the inner and outer boundaries tend to arcs of circles. A similar flow (Hartmann-type) bounded by logarithmic spirals is discussed in [2].

We examine the flow against the line $\mu_1 = \mu_2$ and bounded by the lines A_4A_5, A_4A_2, and $x_1 = x_1^0$ = const (Figs. 1 and 3). For definiteness we choose x_1^0 so that in the plane Ox_1x_2 the straight line C_5C_8 is normal

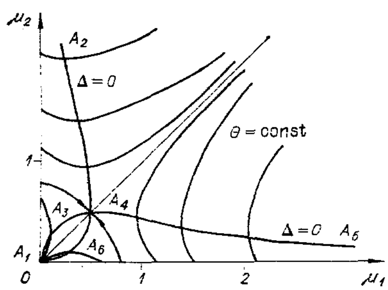

Fig. 1

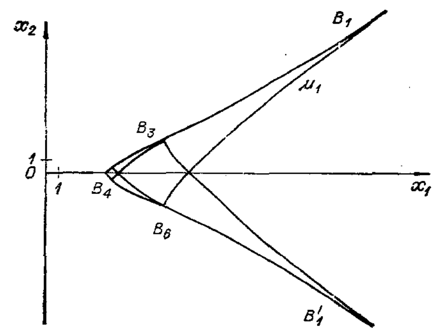

Fig. 2

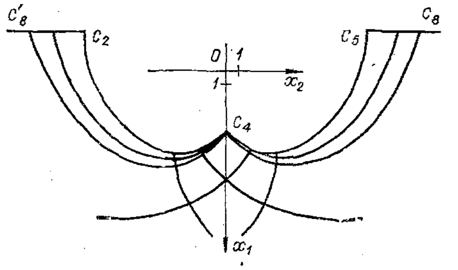

Fig. 3

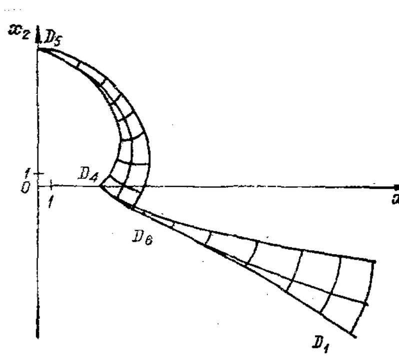

Fig. 4

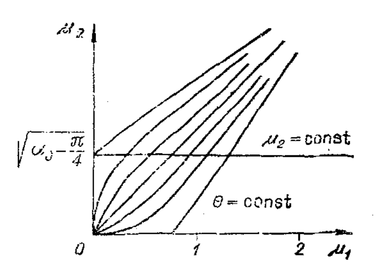

Fig. 5

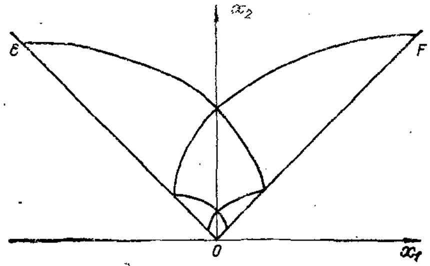

Fig. 6

to the curve C_5C_4. It can be shown that $\cos 2\nu = {}^3/_2\pi \tan \nu$ at the point C_5. The boundary C_4C_5 is acted upon by a constant tangential stress, on the surface C_5C_8 as $x_2 \to \infty$ the tangential stress decays rapidly ($\delta \to \pi/4$), and the normal stress increases ($\rho \to \infty$). Directly against the boundary C_4C_5 the flow has a horn configuration.

Figure 4 shows the slip lines of the flow corresponding to the domain $A_5A_4A_6A_1$ in Fig. 1. The boundaries D_4D_5 and $D_4D_6D_1$ are the envelopes of the families of slip lines μ_1, μ_2. The flow against the boundary D_4D_5 has a horn configuration, and against the boundary $D_4D_6D_1$ a wedge configuration.

Next we consider the flows bounded in the (μ_1, μ_2) plane by the curves A_4A_5, $\theta = \theta^0$, $\theta = \theta^0 + \psi^0$, $|\psi^0| < 2\pi$ and the axis $\mu_2 = 0$. As $\theta^0 \to \infty$ the inner natural boundary of the flow and all the slip lines μ_1 tend to circles, while the slip lines μ_2 tend to the radii $\theta = \text{const}$. If the flow in (μ_1, μ_2) is bounded by the curves $\theta = \theta^0$, $\theta = \theta^0 + \psi^0$ and the condition $\rho^2 \geq {}^1/_2 \cot \nu$, then as $\theta^0 \to \infty$ the inner boundary of the flow tends to a circle, and both families of slip lines tend to logarithmic spirals as $r \to \infty$.

Analogous flows are created in the mapping of domains $\mu_1 \leq \mu_2$.

3. We now take the lower signs in the solution (1.4). In this case there are not real branch lines, and any parameters for which the increment of the angle θ is less than 2π can be taken in Eqs. (1.4). In the (μ_1, μ_2) plane we construct the lines $\theta = \theta^0 = \pi/2 - \alpha_0$. It can be shown that in the (μ_1, μ_2) plane the lines $\pm\alpha_0$ are symmetric about the bisector $\mu_1 = \mu_2$. If $|\alpha_0| \leq \pi/4$, then the line $\theta = \theta^0$ passes through the origin $\mu_1 = \mu_2 = 0$, and the slope angle of the line at the origin is equal to $\pi/4 + \alpha_0$. If $|\alpha_0| \geq \pi/4$, then the lines $\theta = \theta^0$ begin at the points $\rho = \sqrt{|\alpha_0| - \pi/4}$, $\nu = 0$ for $\alpha_0 < 0$ and $\nu = \pi/2$ for $\alpha_0 > 0$. As $\rho \to \infty$ all the lines $\theta = \theta^0$ tend to the bisector $\mu_1 = \mu_2$ (Fig. 5).

The domain $|\alpha_0| \leq \pi/4$ maps onto the quarter-plane $\mathscr{E}OF$ in Fig. 6. As $x_2 \to \infty$ the stressed state inside the quarter-plane tends to uniform, the slip lines tend to the straight lines $x_2 = \pm x_1 + \text{const}$, $\varphi \to \pi/2$, and $\sigma \to 0$. The remaining domains between the curves $\theta = \text{const}$ map onto corresponding wedges in the plane Ox_1x_2. For $|\alpha_0| > \pi/4$, $\theta = \theta^0$, $r \to \infty$ the compression $\sigma/2k$ tends to the finite value $|\alpha_0| - \pi/4$, and the angle $\varphi \to \theta^0 + \pi/4$ for $\alpha_0 > 0$ and $\varphi \to \theta^0 - \pi/4$ for $\alpha_0 < 0$. As $r \to 0$ for any θ^0 the slip lines tend to logarithmic spirals $\delta = -\pi/4$, $\varphi \to \theta^0$, $\sigma \to \infty$. The slip lines $\mu_2 = \mu_2^0 = \text{const}$ tend as $\nu \to 0$ to the origin $x_1 = x_2 = 0$, at which time the angle $\theta \to \infty$. If $\nu \to \pi/2$, the slip line asymptotically approaches the straight line $\theta = (\pi/4) - (\mu_2^0)^2$. The origin $x_1 = x_2 = 0$ is a singularity in the given solution.

4. If necessary, the flow domains (1.4) can be bounded by the introduction of rigid zones or bounding surfaces. Singularities are excluded from the solutions in exactly the same way.

Corresponding to the stresses (1.4) is a certain distribution of rates. We assume that the material is incompressible and that the stress and strain rate tensors are coaxial. Inasmuch as the stresses are known from (1.4), we can reduce the rate equations to a system of telegraphic equations in a plane whose mapping onto the physical plane is known. Those equations can be analyzed by the usual methods on the basis of the Riemann function [1].

We conclude by writing out the original coordinate system the equations for the slip lines and expressions for the stresses σ_{11}, σ_{22}, σ_{12}. In the characteristic coordinates μ_1, μ_2 the solution (1.4) takes the form

$$\frac{\sigma}{2k} = \mu_1^2 + \mu_2^2; \quad \varphi = \mu_1^2 - \mu_2^2 + \frac{\pi}{2}; \tag{4.1}$$

$$r = \frac{\sqrt{\mu_1^2 + \mu_2^2}}{\mu_1\mu_2} e^{\pm 2\mu_1\mu_2}; \quad \theta = \mu_1^2 - \mu_2^2 \mp \operatorname{arctg}\frac{\mu_1}{\mu_2} + \frac{\pi}{4}. \tag{4.2}$$

For μ_2=const (μ_1=const) and a variable μ_1 (μ_2) the radius vector determined by Eqs. (4.2) describe in the plane Ox_1x_2 one of the slip lines of the family μ_1 (μ_2). The corresponding stresses in this case can be calculated by the tensor projection rules and Eqs. (4.1):

$$\sigma_{11} = \sigma + k\cos 2\varphi = 2k(\mu_1^2 + \mu_2^2) - k\cos 2(\mu_1^2 - \mu_2^2);$$
$$\sigma_{22} = \sigma - k\cos 2\varphi = 2k(\mu_1^2 + \mu_2^2) + k\cos 2(\mu_1^2 - \mu_2^2);$$
$$\sigma_{12} = k\sin 2\varphi = -k\sin 2(\mu_1^2 - \mu_2^2).$$

The author wishes to thank E. I. Shemyakin for valuable comments.

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